Generalized Fokker-Planck Equation For Multichannel Disordered Quantum Conductors

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Abstract

The Dorokhov-Mello-Pereyra-Kumar (DMPK) equation, which describes the distribution of transmission eigenvalues of multichannel disordered conductors, has been enormously successful in describing a variety of detailed transport properties of mesoscopic wires. However, it is limited to the regime of quasi one dimension only. We derive a one parameter generalization of the DMPK equation, which should broaden the scope of the equation beyond the limit of quasi one dimension.

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Quantum transport in a disordered N-channel mesoscopic conductor can be described in the scattering approach, initiated by Landauer [1], in terms of the joint probability distribution of the transfer matrices [2,3]. Under very general conditions based on the symmetry properties of the transfer matrices and within the random matrix theory framework [3], the joint probability density of the transmission eigenvalues can be expressed as an evolution with increasing length of the system according to a Fokker-Planck equation known as the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation [4,5]. Such a random matrix approach has been found to be very useful in our understanding of the universal properties in a wide variety of physical systems in condensed matter as well as nuclear and particle physics [6]. In particular, the DMPK equation has been shown to be equivalent [7,8] to the description of a disordered conductor in terms of a non-linear sigma model [9] obtained from the microscopic tight binding Anderson Hamiltonian for non interacting electrons, and is consistent with perturbative calculations and experiments [10,3,11]. The equation has been solved exactly [12], and level correlation functions can be obtained [7] using the method of biorthogonal functions [13]. Because it is extremely difficult to evaluate any higher order correlation function in the sigma model approach, the DMPK equation is more suitable to study the conductance distribution in mesoscopic systems. In recent years it has been applied to a variety of physical phenomena, including conductance fluctuations, weak localization, Coulomb blockade, sub-Poissonian shot noise, etc. [11]. One major limitation of the DMPK equation however is that it is valid only in the regime of quasi one dimension (1D), where the length of the system is much larger than its width [14,11]. While the dependence on geometry of some of the transport properties have been obtained perturbatively [15] in the metallic regime, only limited progress have been made on the extension of the DMPK equation to higher dimensions [16,17]. Currently, there exists no theory for the statistics of transmission levels for all strengths of disorder beyond quasi 1D. This is a particularly severe shortcoming; the important question of the nature of the expected novel kind of universality of the distribution of conductance near the metal-insulator transition [18,19] can not be studied within the powerful DMPK framework, because the transition exists only in higher dimensions.

In this work we argue that the generalization of the DMPK equation to higher dimensions require the relaxation of certain approximations made in the derivation, and suggest a phenomenological way to implement them within the random matrix framework. This allows us to obtain a simple generalization using a phenomenological parameter and the conservation of the total probability. We obtain corrections to the mean and variance of conductance as a function of the parameter using the generalized equation and discuss the implications of the results. We argue that the generalized equation should be valid beyond quasi one dimension.

In the scattering approach, the conductor of length L is placed between two perfect leads of finite width. The scattering states at the Fermi energy define N channels. The $2N \times 2N$ transfer matrix M relates the flux amplitudes on the right of the system to that on the left [2,3]. Flux conservation and time reversal symmetry (in this paper, for simplicity, we will restrict ourselves to the case of unbroken time reversal symmetry only) restricts the number of independent parameters of M to $N(2N + 1)$, and can be represented as [4]

$$M = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \sqrt{1+\lambda} & \sqrt{\lambda} \\ \sqrt{\lambda} & \sqrt{1+\lambda} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}, \tag{1}$$

where u, v are $N \times N$ unitary matrices, and λ is a diagonal matrix, with positive elements $\lambda_i, i = 1, 2 \ldots N$. The physically observable conductance of the system is given by $g = \sum_i (1 + \lambda_i)^{-1}$. Thus the distribution of conductance can be obtained from the distribution of the variables λ_i.

In order to understand the nature of the approximation used in DMPK and to motivate our generalization, we will briefly review the derivation of DMPK following ref. [4]. In this approach, an ensemble of random conductors of macroscopic length $L \gg l$, where l is the mean free path, is described by an ensemble of M random matrices, whose differential probability depends parametrically on L and can be written as $dP_L(M) = p_L(M)d\mu(M)$. Here $d\mu(M)$ is the invariant Haar measure of the group, given in terms of the parameters in (1) by

$$d\mu(M) = J(\lambda) \left[\prod_i^N d\lambda_i \right] d\mu(u)d\mu(v), \tag{2}$$

where

$$J(\lambda) = \prod_{i<j} |\lambda_i - \lambda_j| \tag{3}$$

and $d\mu(u)$ and $d\mu(v)$ are the invariant measures of the unitary group $U(N)$. When a conductor of length L_0 described by a transfer matrix M_0 is added to a conductor of length L_1 and transfer matrix M_1 to form a conductor of length $L = L_1 + L_0$ and transfer matrix $M = M_0 M_1$, the probability density $p_L(M)$ satisfies the combination rule

$$\langle p_{L_1+L_0}(M) \rangle_{L_0} = \int p_{L_1}(M M_0^{-1}) p_{L_0}(M_0) d\mu(M_0), \tag{4}$$

where the angular bracket represensts an ensemble average. For $L_0 \ll l$, the small change in the transfer matrix can be expected to lead to a small change in the parameters λ_i, and one can expand the probability density as

$$\langle p_{L_1+L_0}(\lambda) \rangle_{L_0} = \langle p_{L_1}(\lambda + \delta\lambda) \rangle_{L_0} = p_{L_1}(\lambda) + \sum_a \frac{\partial p_{L_1}(\lambda)}{\partial \lambda_a} \langle \delta\lambda_a \rangle_{L_0} + \frac{1}{2} \sum_{ab} \frac{\partial^2 p_{L_1}(\lambda)}{\partial \lambda_a \partial \lambda_b} \langle \delta\lambda_a \delta\lambda_b \rangle_{L_0}. \tag{5}$$

Since the changes in λ_a are small, we can use perturbation theory to evaluate their averages. We can also expand the left hand side in powers of L_0. The resulting equation, keeping only terms first order in L_0 on the left hand side, is given by

$$L_0 \frac{\partial p}{\partial L} = \sum_a (1 + 2\lambda_a) \frac{\partial p}{\partial \lambda_a} \left\langle \sum_c \lambda_c' v_{ca}'^* v_{ca}' \right\rangle_{L_0} + \sum_a \lambda_a (1 + \lambda_a) \frac{\partial^2 p}{\partial \lambda_a^2} \left\langle \sum_c \lambda_c'(1 + \lambda_c')|v_{ca}'|^4 \right\rangle_{L_0}$$

$$+ \sum_{a \neq b} \frac{\lambda_a + \lambda_b + 2\lambda_a \lambda_b}{\lambda_a - \lambda_b} \frac{\partial p}{\partial \lambda_a} \left\langle \sum_c \lambda_c'(1 + \lambda_c') v_{ca}'^* v_{cb}'^* v_{cb}' v_{ca}' \right\rangle_{L_0}. \tag{6}$$

Here the primed variables correspond to the added small conductor of length L_0.

The above equation (6) is quite general. It is based on the symmetry properties of the transfer matrices and the combination principle for adding two conductors. These principles should remain valid beyond quasi one dimension. It is the further approximations on the averages in equation (6) made in deriving DMPK that limits DMPK to quasi one dimension. There are two major approximations involved:

(i) The 'isotropy' assumption is used to *decouple* the averages over the products of the parameters λ and the unitary matrices v. Once decoupled, the averages over the products of the unitary matrices alone can be explicitly obtained to give

$$\left\langle v_{ca}'^* v_{ca}' \right\rangle = \frac{1}{N}; \quad \left\langle v_{ca}'^* v_{cb}'^* v_{cb}' v_{ca}' \right\rangle = \frac{1}{N(N+1)}; \quad \left\langle |v_{ca}'|^4 \right\rangle = \frac{2}{N(N+1)}, \tag{7}$$

while the average over the trace of λ_c' is taken to be proportional to L_0. In particular, $\langle \sum_c \lambda_c' \rangle = N L_0/l$, where l is the mean free path, consistent with the Born approximation for the transmission amplitude valid for small L_0.

(ii) The second approximation is based on the expectation that the averages of the products of λ_c' are higher orders in L_0, and therefore negligible. In particular, this means that the terms proportional to $\sum_c \lambda_c'^2$ are neglected in equation (6).

The above two approximations, together with the identity

$$\sum_{b(\neq a)} \frac{\lambda_a + \lambda_b + 2\lambda_a\lambda_b}{\lambda_a - \lambda_b} = -(N-1)(1 + 2\lambda_a) + 2\lambda_a(1 + \lambda_a) \sum_{b(\neq a)} \frac{1}{\lambda_a - \lambda_b}, \tag{8}$$

lead to the well known DMPK equation:

$$\frac{\partial p}{\partial (L/l)} = \frac{2}{N+1} \frac{1}{J(\lambda)} \sum_a \frac{\partial}{\partial \lambda_a} \left[\lambda_a (1 + \lambda_a) J(\lambda) \frac{\partial p(\lambda)}{\partial \lambda_a} \right], \tag{9}$$

where $J(\lambda)$ is defined in (3).

We will first show that beyond quasi one dimension, the second approximation fails, namely $\sum_c \lambda_c'^2$ is of the same order in L_0 as $\sum_c \lambda_c'$ and therefore can not be neglected. In this case we will show that the total probability can not be conserved within the decoupling approximation. We will then introduce phenomenological parameters for the averages over the products in (6), and show that the conservation of total probability require a very specific generalization of the DMPK equation involving a single additional parameter. Finally we will evaluate the corrections to the mean and variance of the conductance using the generalized DMPK as a function of the parameter and interpret the results.

To go beyond quasi 1D, we start with a conductor of length L_0 along x and width W along y and z, with scattering potential $V(x, y, z)$. To see how the second approximation fails, we will consider, for simplicity, a square well potential adequately approximated by a repulsive delta function at $x = 0$, i.e $V(x, y, z) = V_T(y, z)\delta(x)$. Writing the Schrödinger wavefunction as $\Psi(x, y, z) = \sum_i \psi_i(y, z)\phi_i(x)$, where $\psi_i(y, z)$ are the transverse eigenfunctions in the perfectly conducting lead, chosen to be real, we obtain the system of coupled equations for the N channels

$$\phi_i''(x) + k_i^2 \phi_i(x) = \sum_i \kappa_{ij}(x)\phi_j(x), \tag{10}$$

4

where the prime denotes a derivative with respect to x, k_i are the wavevectors in channel i, and κ_{ij} are the coupling constants given by $\kappa_{ij}(x) = (2m/\hbar) \int \int dy dz \psi_j(y,z) V_T(y,z) \psi_i(y,z)$. We are interested in the transfer matrix M that connects the solution ϕ on the left side of the conductor with that on the right side. The transfer matrix satisfying the flux conservation and time reversal symmetry can be written in the form

$$M = \begin{pmatrix} \mathbf{1} + \Delta & \Delta \\ \Delta^* & \mathbf{1} + \Delta^* \end{pmatrix}, \tag{11}$$

where $\mathbf{1}$ and Δ are $N \times N$ matrices and $\Delta_{ij} = \kappa_{ij}/2ik_i$. Note that Δ is pure imaginary but not symmetric. The parameters λ that satisfy the DMPK equation in quasi 1D are the eigenvalues of the matrix $X = [Q + Q^{-1} - 2 \cdot \mathbf{1}]/4$, where $Q = M^\dagger M$ [3]. From flux conservation, $Q^{-1} = \Sigma_z Q \Sigma_z$ where Σ_z is the third Pauli matrix with 1 and 0 replaced by $(N \times N)$ $\mathbf{1}$ and $\mathbf{0}$ matrices. It is easy to see that X is block diagonal, each block given by a sum of two matrices $X_1 = (\Delta + \Delta^\dagger)/2$ and $X_2 = \Delta^\dagger \Delta$. The important point is that X_1 is traceless, so $\mathrm{tr}(\lambda_i)$ is given by $\mathrm{tr}(X_2) = \mathrm{tr}(\Delta^\dagger \Delta)$. On the other hand, X_1 does contribute to $\mathrm{tr}(\lambda_i^2) = \mathrm{tr}(X_1 + X_2)^2$, where $\mathrm{tr}(X_1)^2 = \mathrm{tr}(\Delta^2 + \Delta^{\dagger 2} + \Delta^\dagger \Delta + \Delta \Delta^\dagger)/4$. Clearly it is of the same order as $\mathrm{tr}(\lambda_i)$, and can not be neglected.

It is now straightforward to show that keeping the $\mathrm{tr}(\lambda_i^2)$ terms in (6) and using the decoupling approximation of the averages of v and λ lead to a breakdown of the conservation of total probability. Suppose $\left\langle \sum_c \lambda_c'^2 \right\rangle = \alpha L_0/l$. Then using (7) for the averages over v, we get a correction term to the DMPK equation equal to

$$-\frac{\alpha}{2} \sum_a (1 + 2\lambda_a) \frac{\partial p}{\partial \lambda_a}. \tag{12}$$

Clearly this is not a sum of total derivatives and the resulting equation does not conserve total probability [20].

It is therefore clear that in order to go beyond quasi 1D, we need to relax both approximations. We propose a simple phenomenological way to take care of both. Instead of computing the three averages in (6) explicitly, we start with the following very general ansatz:

$$\left\langle \sum_c \lambda_c' v_{ca}'^* v_{ca}' \right\rangle_{L_0} = \frac{L_0}{l}; \quad \left\langle \sum_c \lambda_c'(1 + \lambda_c') v_{ca}'^* v_{cb}'^* v_{cb}' v_{ca}' \right\rangle_{L_0} = \frac{L_0}{l} \frac{1}{N+1} \mu_1;$$

$$\left\langle \sum_c \lambda_c'(1 + \lambda_c') |v_{ca}'|^4 \right\rangle_{L_0} = \frac{L_0}{l} \frac{2}{N+1} \mu_2 \tag{13}$$

where μ_1 and μ_2 are arbitrary dimensionless parameters, which can be functions of N. Clearly, $\mu_1 = \mu_2 = 1$ gives back the quasi 1D limit. Note that any additional parameter in the first term will only serve to redefine the mean free path, so there are only two additional parameters possible. With this ansatz, equation (6) becomes

$$\frac{\partial p}{\partial(L/l)} = (1 - \mu_1 \frac{N-1}{N+1}) \sum_a (1 + 2\lambda_a) \frac{\partial p}{\partial \lambda_a} + \frac{2\mu_2}{N+1} \sum_a \lambda_a(1 + \lambda_a) \frac{\partial^2 p}{\partial \lambda_a^2}$$

$$+ \frac{2\mu_1}{N+1} \sum_a \lambda_a(1 + \lambda_a) \frac{1}{J} \frac{\partial J}{\partial \lambda_a} \frac{\partial p}{\partial \lambda_a}. \tag{14}$$

We now demand that the parameters μ_1 and μ_2 are such that the right hand side can be written as a sum of total derivatives in order to ensure the conservation of total probability. Note that the special choice $\mu_1 = \mu_2 = 1$ makes the coefficients of all three terms on the right hand side of (14) the same, and then the three terms can be written as a sum of derivatives after multiplying by $J(\lambda)$. It may appear at first that with two parameters and three terms, no other choice is possible, except for a trivial multiplicative factor for all three terms which can be absorbed in the redefinition of the mean free path. However, we note that if we choose

$$(1 - \mu_1 \frac{N-1}{N+1}) = \frac{2\mu_2}{N+1}, \tag{15}$$

together with a renormalization of the measure

$$J \to J^\gamma; \quad \gamma = \frac{\mu_1}{\mu_2}, \tag{16}$$

then (14) can be rewritten as

$$\frac{\partial p}{\partial (L/l')} = \frac{2}{N+1} \frac{1}{J^\gamma(\lambda)} \sum_a \frac{\partial}{\partial \lambda_a} \left[\lambda_a (1 + \lambda_a) J^\gamma(\lambda) \frac{\partial p(\lambda)}{\partial \lambda_a} \right], \tag{17}$$

where $l' = l/\mu_2$ is a renormalized mean free path. Equation (17) is our one parameter generalization of the DMPK equation (9), where the parameter γ enters in the renormalization of the measure as in (16). Note that in the absence of time reversal symmetry or in the presence of spin-orbit scattering, the measure is changed in a similar way by an exponent $\beta = 2, 4$ respectively [14,21]. However, in our present case with time reversal symmetry, $\beta = 1$, and the exponent γ is in general non integral. Clearly $\gamma = 1$ is the quasi 1D limit. From the relation between μ_1 and μ_2, and the condition that both μ_1 and μ_2 must be positive, we find the following restrictions:

$$0 < \mu_1 < \frac{N+1}{N-1}; \quad 0 < \mu_2 < \frac{N+1}{2}. \tag{18}$$

This means that the only restriction on the parameter γ is that it is positive. In general, it can be a function of N.

We can try to interpret the phenomelogical parameter γ by comparing with known results. The expectation value of any function $F(\lambda)$, defined as

$$\langle F \rangle_{(L/l')} = \int F(\lambda) p_{(L/l')}(\lambda) J^\gamma(\lambda) \prod_{a=1}^N d\lambda_a, \tag{19}$$

follows an evolution equation which can be obtained by multiplying both sides of (17) by $J^\gamma(\lambda) F(\lambda)$ and integrating over all λ_a, giving

$$\frac{\partial \langle F \rangle_s}{\partial s} = \left\langle \sum_a \left[(1 + 2\lambda_a) \frac{\partial F}{\partial \lambda_a} + \lambda_a (1 + \lambda_a) \frac{\partial^2 F}{\partial \lambda_a^2} \right] + \frac{\gamma}{2} \sum_{a \neq b} \frac{\lambda_a (1 + \lambda_a) \frac{\partial F}{\partial \lambda_a} - \lambda_b (1 + \lambda_b) \frac{\partial F}{\partial \lambda_b}}{\lambda_a - \lambda_b} \right\rangle, \tag{20}$$

where $s = L/l'$. If γ is independent of N, then we can use the method of moments in [14] to obtain the average and variance of the conductance $g = \sum_i (1 + \lambda_i)^{-1}$ as a power series in $s/N \ll 1$ in the large N and large s limit. We find that to leading order, $\langle g \rangle = Nl'/L - (2 - \gamma)/3\gamma$, and $\text{var}(g) = \langle g^2 \rangle - \langle g \rangle^2 = 2/15\gamma$. As expected, $\gamma = 1$ gives back the quasi 1D results. However, in general the variance decreases with increasing γ. Comparing with the result $\text{var}(g) \sim \sqrt{L_y L_z}/L$ for a rectangular conductor with length $L_x = L$ and cross-section $L_y L_z$ [22], we see that the parameter γ can be identified with the aspect ratio $L/\sqrt{L_y L_z}$ in this diffusive transport regime.

If $\gamma \sim 1/N$, obtained for $\mu_2 = \nu N$, which is consistent with the restriction (18) if $\nu < 1/2$, then we need to divide both sides of (20) by γ, so that the renormalized mean free path $l'' = l'/\gamma = l/\mu_1$ is independent of N. Then assuming $\nu \ll 1$, it is possible to obtain corrections to the $1/N$ expansion up to linear order in ν using the above method of moments. The result is that the corrections are larger by a factor $\nu \mu_1 s$, which signals the breakdown of the expansion in the large s limit. It would be interesting to obtain a more rigorous solution of (17) for arbitrary γ.

In summary, by relaxing certain approximations in the derivation of the DMPK equation (9) which limits it to the quasi 1D regime only, we have derived a one parameter generalization given in eq. (17), based on a phenomenological ansatz and the conservation of total probability. The geometry dependence of the parameter, obtained in the diffusive limit by evaluating the correction to the variance of the conductance beyond its quasi 1D value, suggests that the generalized equation should be applicable beyond the quasi 1D regime. This should broaden the scope of the DMPK approach.

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